MANDURAH CATHOLIC COLLEGE

Complex numbers
Test 1 2019
Section One – Calculator Free

MATHEMATICS Specialist Unit 3
Year 12

NAME: Solutions.

TEACHER: ______________________________

CALCULATOR FREE:______/26   CALCULATOR ASSUMED______/24

TOTAL: ______/50   PERCENTAGE: ______ %

TIME ALLOWED FOR THIS PAPER

Working time for paper:
Section One: 25 minutes
Section Two: 25 minutes
Total Time = 50 minutes

MATERIALS REQUIRED/RECOMMENDED FOR THIS PAPER

TO BE PROVIDED BY THE TEACHER

This Question/Answer Booklet
SCSA Formula Sheet

TO BE PROVIDED BY THE STUDENT

Standard Items: Pens, pencils, eraser or correction tape, ruler, protractor.

IMPORTANT NOTE TO STUDENTS

No other items may be taken into the classroom. It is your responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the classroom. If you have any unauthorised material with you, hand it to the teacher BEFORE reading any further.

Instructions to Students

1. **ALL** questions should be attempted.

2. Write your answers in the spaces provided in this Question/Answer Booklet.

3. **SHOW ALL YOUR WORKING CLEARLY.** Your working should be sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Correct answers given without supporting reasoning may not be allocated full marks. Incorrect answers given without supporting reasoning cannot be allocated any marks.

4. If you repeat an answer to any question, ensure that you cancel the answers you do not wish to have marked.

5. It is recommended that you **do not use pencil**, except in diagrams.

Question 1 **[1, 1, 1 = 3 marks]**

For the complex numbers $z = 1 - \sqrt{3}i$ and $w = 2\,cis\left(\frac{\pi}{4}\right)$

(a) Express z in polar form, $r \text{ cis } \theta$, where $-\pi < \theta \leq \pi$.

$2 \text{ cis}\left(\frac{-\pi}{3}\right)$ ✓

(b) Find $\bar{z}w$ expressing your answer in Cartesian form.

$w = \sqrt{2} + i\sqrt{2}$

$\bar{z} = 1 + i\sqrt{3}$

$\therefore \bar{z}w = (1 + i\sqrt{3})(\sqrt{2} + i\sqrt{2})$

$= \sqrt{2} + i\sqrt{2} + i\sqrt{6} - \sqrt{6}$

$= (\sqrt{2} - \sqrt{6}) + i(\sqrt{2} + \sqrt{6})$ ✓

(c) Find $\frac{w}{z}$ expressing your answer in polar form.

$= \dfrac{2 \text{ cis}\left(\frac{\pi}{4}\right)}{2 \text{ cis}\left(\frac{-\pi}{3}\right)}$

$= \text{cis}\left(\frac{7\pi}{12}\right)$ ✓

3

Question 2 [3 marks]

Showing use of De Moivre's theorem, express $\text{Cos } 3\theta$ in terms of $\text{Cos } \theta$.

Let $\cos 3\theta = (\cos\theta + i\sin\theta)^3$

$= \cos^3\theta + 3\cos^2\theta\, i\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

$\therefore \cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$= \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$

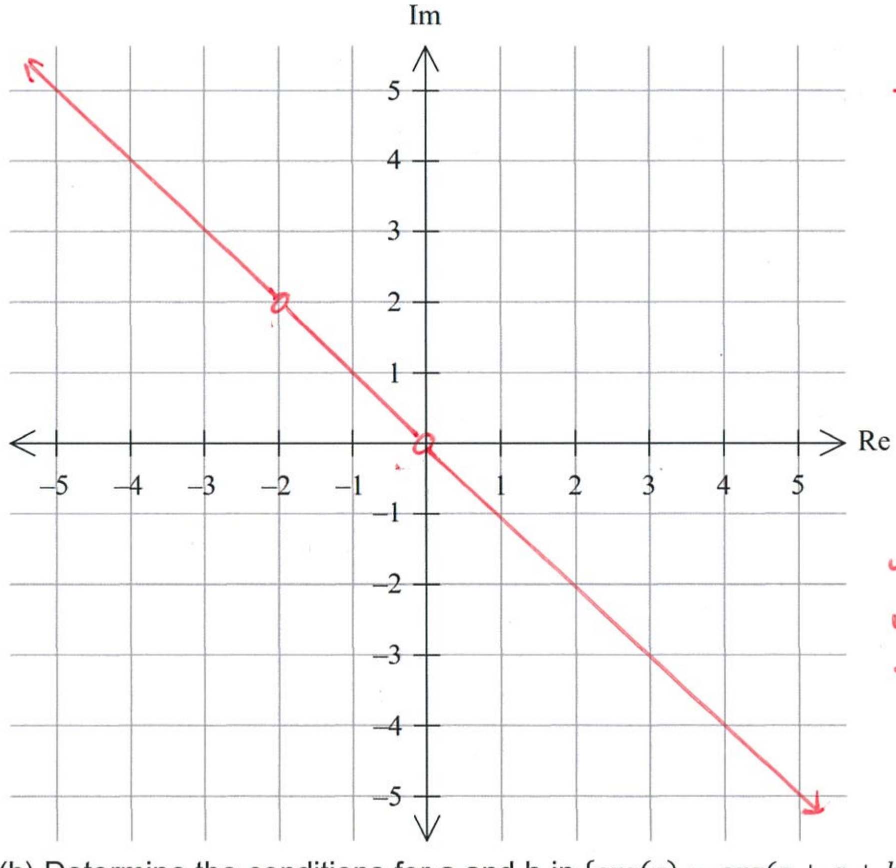
$= \cos^3\theta - 3\cos\theta + 3\cos^3\theta$

$= 4\cos^3\theta - 3\cos\theta$

✓ use of de Moivre's
✓ Expands and equates real
✓ simplifies to $\cos\theta$

Question 3 [3, 2 = 5 marks]

(a) Sketch the solution to $\{\arg(z) = \arg(z + 2 - 2i)\}$



Let $z = x + iy$

$\therefore \arg(x+iy) = \arg(x+iy+2-2i)$

$\tan^{-1}\dfrac{y}{x} = \tan^{-1}\dfrac{y-2}{x+2}$

$\dfrac{y}{x} = \dfrac{y-2}{x+2}$

$yx + 2y = xy - 2x$

$2y = -2x$

$y = -x,\ x \neq 0, x \neq 2$

✓ Shows $(0,0)$ as no sol
✓ Shows $y = -x$ as sol.
✓ Shows no sol for $(-2,0)$

(b) Determine the conditions for a and b in $\{\arg(z) = \arg(z + a + bi)\}$ to produce infinite solutions and explain any gaps in the set of solutions.

For solutions to exist, a, b must conform to $y = -x$ to ensure $\arg(z) = \arg(z + a + bi)$

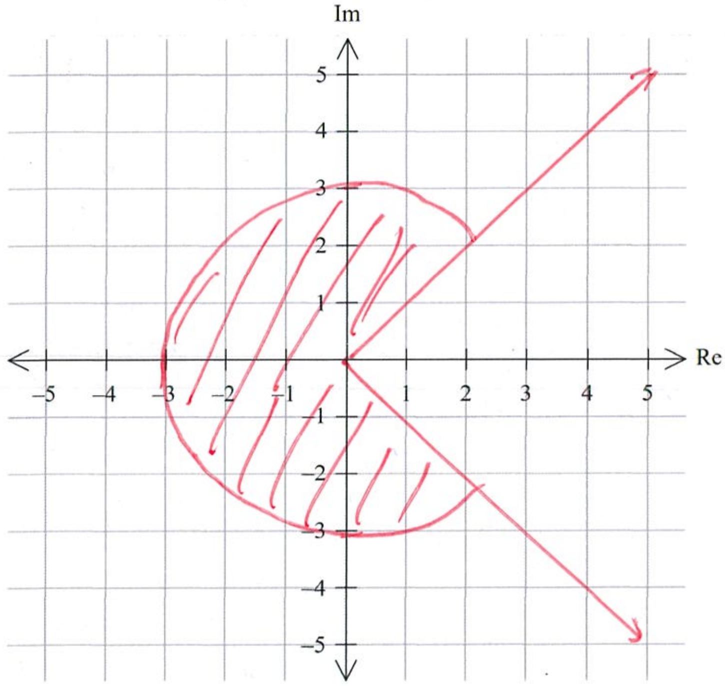
A gap will exist at $(0,0)$ and $(-a,-b)$ ✓✓

8

Question 4 **[3, 4 = 7 marks]**

Sketch and shade the region in the argand plane defined by:

(a) $\{z: \frac{\pi}{4} \leq \arg(z) \leq -\frac{\pi}{4} \cap r \leq 3\}$

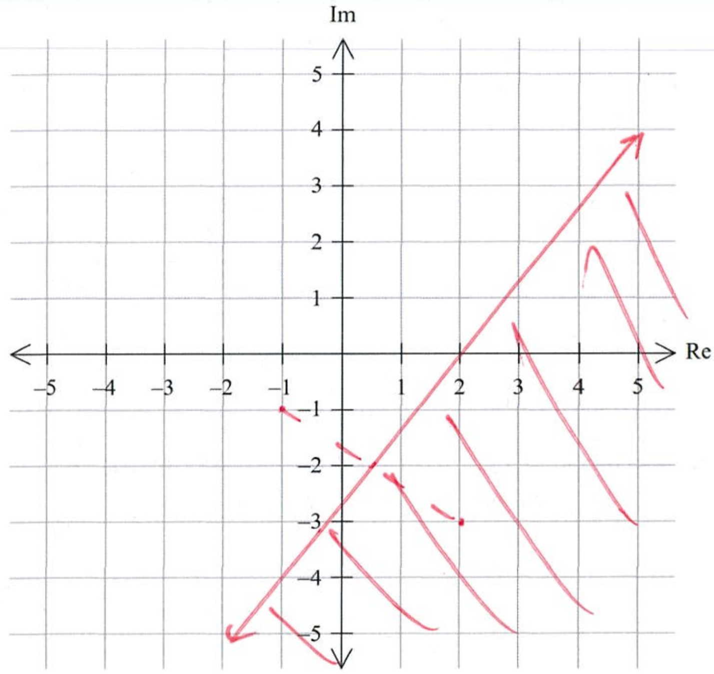


✓ Plots $\arg(z) = \pm\frac{\pi}{4}$

✓ Plots $r = 3$

✓ Shows region

(b) $\{z: |z - 2 + 3i| \leq |z + 1 + i|\}$



✓ Plots points

✓ Line

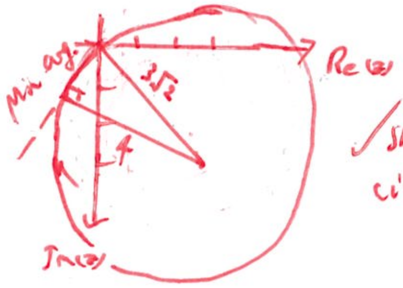
✓ Region

7

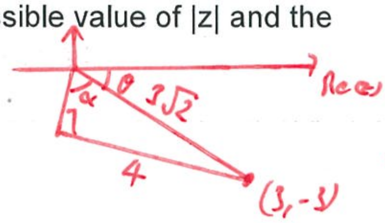
Question 5 **[4 marks]**

For $\{z: |z - 3 + 3i| = 4\}$ determine the minimum possible value of |z| and the minimum arg(z).

$|z - (3 - 3i)| = 4$



✓ Shows circle



✓ Creates Δ.

$\theta = \tan^{-1}\left(\frac{3}{3}\right)$

$\theta = -\frac{\pi}{4}$

$\alpha = \sin^{-1}\left(\frac{4}{3\sqrt{2}}\right)$

$\alpha = -\sin^{-1}\frac{2\sqrt{2}}{3}$

$\therefore$ min $\arg(z) = -\frac{\pi}{4} - \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$ ✓

Min $|z| = 3\sqrt{2} - 4$ units. ✓

Question 6 **[4 marks]**

Given that $x - 1$ is a factor of $F(x) = x^4 - 6x^3 - 3x^2 + 20x - 12$, show that $F(1) = 0$ hence completely factorise $F(x)$.

$F(1) = 1^4 - 6(1)^3 - 3(1)^2 + 20(1) - 12$

$= 0$

✓ Shows $F(1) = 0$

$\therefore F(x) = (x-1)(ax^3 + bx^2 + cx + d)$

$a = 1, \quad b = -5, \quad c = -8, \quad d = 12$

✓ coefficients of cubic

$\therefore F(x) = (x-1)(x^3 - 5x^2 - 8x + 12)$

$x - 1$ is a factor of the cubic

$F(x) = (x-1)(x-1)(ax^2 + bx + c)$

✓ quadratic

$c = -12, \quad a = 1, \quad b = -4$

✓

Hence, $F(x) = (x-1)^2(x-6)(x+2)$

End of Section One

(8)

Additional working space

Question number:__________

MANDURAH CATHOLIC COLLEGE

Complex Numbers
Test 1 2019
Section 2 Calculator-Assumed

MATHEMATICS Specialist Unit 3
Year 12

NAME: ____

TEACHER:____

RESULT CA:_______/24

TIME ALLOWED FOR THIS PAPER

Working time for paper: Section 1 = 25 minutes
Section 2 = 20 minutes
Total Time = 45 minutes

MATERIALS REQUIRED/RECOMMENDED FOR THIS PAPER

TO BE PROVIDED BY THE TEACHER

This Question/Answer Booklet
Formula Sheet

TO BE PROVIDED BY THE STUDENT

Standard Items: Pens, pencils, eraser or correction tape, ruler, protractor.
Special Items: Scientific/CAS calculator, 1 A4 (one sided) page of notes

IMPORTANT NOTE TO STUDENTS

No other items may be taken into the classroom. It is your responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the classroom. If you have any unauthorised material with you, hand it to the teacher BEFORE reading any further.

Instructions to Students

1. **ALL** questions should be attempted.

2. Write your answers in the spaces provided in this Question/Answer Booklet.

3. **SHOW ALL YOUR WORKING CLEARLY.** Your working should be sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Correct answers given without supporting reasoning may not be allocated full marks. Incorrect answers given without supporting reasoning cannot be allocated any marks.

4. If you repeat an answer to any question, ensure that you cancel the answers you do not wish to have marked.

5. It is recommended that you **do not use pencil**, except in diagrams.

Question 7 **[2, 3 = 5 marks]**

(a) Multiplying a non-zero complex number by $\frac{1-i}{1+i}$ results in what rotation about the origin on the Argand plane?

$1-i = \sqrt{2}\,cis\left(-\frac{\pi}{4}\right)$ $\quad\therefore \frac{1-i}{1+i} = cis\left(-\frac{\pi}{2}\right)$

$1+i = \sqrt{2}\,cis\left(\frac{\pi}{4}\right)$

$\therefore$ $90°$ rotation clockwise

✓ Converts and simplifies

✓ Answer

(b) The complex number $z = a\,Cis\theta$ and $w = b\,Cis\alpha$ where $-\pi \le \theta, \alpha \le \pi$ satisfy,

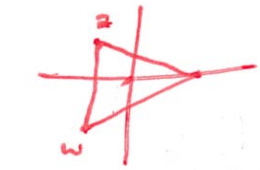
$$1 + z + w = 0$$

1, z and w form the vertices of an equilateral triangle on the Argand plane, determine the coordinates of the vertices.

$1 + z + w = 0$

$z + w = -1$

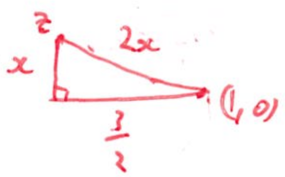
Hence, z, w need to be conjugates.



✓ Diagram

$z, w = -\frac{1}{2}$

$Re(z) = Re(w) = -\frac{1}{2}$



$\therefore 4x^2 = x^2 + \frac{9}{4}$

$3x^2 = \frac{9}{4}$

$x^2 = \frac{3}{4}$

$x = \pm\frac{\sqrt{3}}{2}$

$\therefore$ Vertices are

$(1, 0)$, $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ ✓ One vertex

✓ All vertices.

5

Question 8 **[2, 2, 3 = 7 marks]**

Given $z = r\ cis\ \theta$ is any complex number

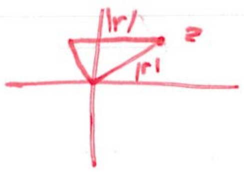
(a) Simplify $2i\bar{z}$ into $r\ cis\ \theta$ format.

$2i\bar{z} = 2\,cis\frac{\pi}{2} \cdot r\,cis\,-\theta$

$= 2r\,cis\left(\frac{\pi}{2}-\theta\right)$ ✓ modulus

✓ argument

(b) Determine the nature of the triangle formed by $z - r$ and explain why $\theta \neq 0, \pi$.



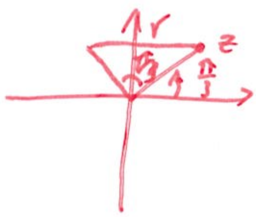
∴ Δ must be isosceles as two sides will have length |r| ✓

When θ = 0, z and r are collinear and the Δ doesn't exist. ✓

(c) Identify the case where the triangle formed is equilateral and determine an equation for the third side, w, which fits z – r + w = 0.

For triangle to be equilateral internal angles must be $\frac{\pi}{3}$ (60°). ✓ identifies angles

∴ If $z = r\,cis\,\theta$, $\theta = \frac{\pi}{3}$ ✓



For $z - r + w = 0$

$z + w = r$

Hence z, w need to be conjugates.

∴ $w = r\,cis\left(-\frac{\pi}{3}\right)$ ✓

7

Question 9 **[3 marks]**

Given that the complex number $z = a + bi$, determine a and b given $Im\left(\frac{2z+i}{z}\right) = 0$ and $Re\left(\frac{2z+i}{z}\right) = 10$.

$$Im\left(\frac{2(a+bi)+i}{a+bi}\right) = 0$$

$$Im\left(\frac{2a+2bi+i}{a+bi}\right) = 0$$

$$0 = Im\left(\frac{2a+i(2b+1)}{a+bi} \times \frac{a-bi}{a-bi}\right)$$ ✓ realises denominator

$$0 = Im\left(\frac{2a^2-2abi+2abi+2b^2+ai+b}{a^2+b^2}\right)$$

$$\therefore \frac{a}{a^2+b^2} = 0$$

$a = 0$ ✓ solves for a

$$Re(z) = \frac{2a^2+2b^2+b}{a^2+b^2}$$

$$10 = \frac{2b^2+b}{b^2}$$

$10b^2 = 2b^2+b$

$8b^2 - b = 0$

$b(8b-1) = 0$ ✓

$b = 0,\ b = \frac{1}{8}$ but $b \neq 0$ as original equations would be undefined.

3

Question 10 **[4, 5 = 9 marks]**

(a) Given $z^3 = -1$ determine all solutions in $r\ cis\ \theta$.

$z^3 = cis(\pi)$ ✓

$z = cis\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right), \quad k \in \mathbb{Z}$ ✓

$z_0 = cis\left(\frac{\pi}{3}\right), \quad z_1 = cis(\pi), \quad z_2 = cis\left(-\frac{\pi}{3}\right)$ ✓one solution ✓All solutions.

(b) Given $P(z) = Q(z).R(z)$ and $Q(z) = z^2 - 2z + 5$ and $R(z) = z^3 + 1$ solve the following equation giving all answers in $a + bi;\ -\pi \leq \theta \leq \pi$

$$T(z) = z^5 - 2z^4 + 5z^3 + z^2 - 2z + 5$$

$T(z) = P(z) = Q(z).R(z)$ ✓recognises link

$\therefore (z^2 - 2z + 5)(z^3 + 1) = 0$ ✓uses Null Factor Law.

(1) $z^3 = -1$

$z_0 = \frac{1+\sqrt{3}i}{2}$

$z_1 = -1$ ✓

$z_2 = \frac{1-\sqrt{3}i}{2}$

(2) $z^2 - 2z + 5 = 0$ ✓

Using CAS

$z = 1 \pm 2i$ ✓

End of Assessment

9

Additional working space

Question number:__________